[2] Hoff, N. J., "Bending and Buckling of Rectangular Sandwich Plates," TN 2225, 1950, NACA.

[3] Kan, Han-pin and Huang, Ju-chin, "Large Deflection of Rectangular Sandwich Plates," *AIAA Journal*, Vol. 5, No. 9, Sept. 1967, pp. 1706–1708.

Comments on "Natural Vibrations of a Stiffened Pressurized Cylinder with an Attached Mass"

ROBERT DAN SMITH*
Lockheed Palo Alto Research Facility, Palo Alto, Calif.

THE following comments concern the article by I. U. Ojalvo and M. Newman[1] which appeared in the June 1967 issue of the *AIAA Journal*. In this article the authors investigate theoretically and experimentally the free vibrations of an internally pressurized, cylindrical shell to which is attached a concentrated mass. In a number of cases the shell is stiffened by the addition of rings that subdivide the shell into bays. In the remaining cases the shell is further stiffened by the addition of stringers.

First, the applicability of the assumption of zero axial restraint, for the edges of the bays investigated in the analysis, is open to question. Each of the intermediate bays of the shell is continuously connected to the two adjacent bays. This situation is not altered by the attachment, at discrete rivet locations, of rings that have high out-of-plane twisting compliances. If an intermediate bay is excited in a frequency range where the radial motion of the shell is dominant in comparison to the axial motion, as in the experiments, then $|u|]_{\text{adjacent bays}} \ll |u|]_{\text{excited bay}}$ since $|w|]_{\text{adjacent bays}} \ll |w|]_{\text{excited bay}}$. This indicates the presence of appreciable axial forces at the edges of the excited bay, which is investigated in the analysis.

In addition, upon selecting an analytical model of the coupled shell-attached mass system involving a shell having axially unrestrained edges, the authors do not state whether a rigid body mode[2] of the form $(u, v, w) = (1, 0, 0)$ has or has not been included in the modal expansion given for the shell displacements. Since, in the analysis, the shell is axially unrestrained, such a mode is necessary in order to allow a nonzero acceleration of the mass center of the shell, due to the axial force exerted by the attached mass. It should be noted, however, that this rigid body mode would not be present in the axial motion of the shell for the forced response problem, associated with the experiments, in which the analytical model has both the attached mass and the radial exciting force located at $x = L/2$.

Second, consider, as the authors do, only the case for which radial shell motion is symmetric about $\theta = 0$. Now there exists a one-to-one correspondence between the frequencies for the unweighted modes and those for the coupled system modes, i.e., the addition of the attached mass does not create any new frequencies but causes a downward perturbation of the unweighted system frequencies. In addition, for the case in which the forces exerted by the attached mass do not act through displacements of zero amplitude, no coupled system frequency may be equal to an unweighted system frequency. If this were not true, the attached mass, oscillating in a coupled system mode, could drive at resonance one of the unweighted shell modes. A comparison of Tables 2 and 3, for cylinder 1, reveals that there exist two coupled system frequencies below the lowest unweighted system frequency in apparent contradiction to the preceding statements. A possible interpretation of this result is that one of the set of coupled system modes, involving antisymmetric radial motion of the shell about $\theta = 0$, has been excited. Such modes have not been considered by the authors. A typical mode of this group involves motion of the attached mass in the circumferential direction as well as shell displacements which can be described by an expansion in the unweighted antisymmetric modal displacements[3] about $\theta = 0$, i.e.,

$$u = q_{kl}^{(1)} \sin(k\theta) \cos(\lambda_l x) \qquad v = -q_{kl}^{(2)} \cos(k\theta) \sin(\lambda_l x)$$

$$w = q_{kl}^{(3)} \sin(k\theta) \sin(\lambda_l x)$$

where $k = 0, 1, 2, 3, \ldots$ and $l = 1, 2, 3, \ldots$ are possible choices for k and l.

Third, for the forced response problem associated with the experiments, i.e., the attached mass and the radial exciting force located at $\bar{x} = L/2$, the axial displacement of the attached mass is equal to zero. However, for the general case of free vibration of the coupled system and $\bar{x} \neq L/2$, it is to be noted that as $\bar{n}$, the number of unweighted modes used in the analysis of the coupled system, becomes infinite, i.e., as one hopefully proceeds toward an exact solution, the axial displacement of the point mass

$$[u]_{\substack{\theta=0\\x=\bar{x}}}$$

given in Eq. (29) of Ref. 1 becomes infinite. Thus the kinetic energy of the system, as given in Eq. (28) of Ref. 1, also becomes infinite. Such difficulties tend to cast doubt on the validity of the analysis. The divergence of the modal representation for

$$[u]_{\substack{\theta=0\\x=\bar{x}}}$$

may be seen from the following considerations. Let $\tilde{u}$ be the axial displacement of the unweighted shell at $x = \bar{x}$, $\theta = 0$ due to an axial unit harmonic point force $e^{i\omega_c t}$. In addition, let the index j denote the free vibration modes of the unweighted shell. Utilizing the notation of Ref. 1, $\tilde{u}$ can be described in terms of these modes as

$$\tilde{u} = -\frac{1}{M_0\omega_c^2} + \sum_{j=1}^{\infty} \frac{[Q_j^{(1)} \cos(\lambda_j \bar{x})]^2}{M_j(\omega_j^2 - \omega_c^2)}$$

where M_0 is the mass of the shell and M_j is the generalized mass for mode j. The admittance quantity $\tilde{u}$ may be related to

$$[u]_{\substack{\theta=0\\x=\bar{x}}}$$

via the equation

$$[u]_{\substack{\theta=0\\x=\bar{x}}} = \tilde{u} \text{ (axial force on the cylinder due to the point mass)}$$

Let us also consider the static problem of the same cylinder subjected to an axial point force, of unit magnitude, located $x = \bar{x}$, $\theta = 0$ and counterbalanced by a uniform body force distribution. Utilizing the modes of the unweighted shell, one can express the axial displacement $\hat{u}$ at $x = \bar{x}$, $\theta = 0$ as

$$\hat{u} = \sum_{j=1}^{\infty} \frac{[Q_j^{(1)} \cos(\lambda_j \bar{x})]^2}{M_j\omega_j^2}$$

One can also represent $\hat{u}$ in terms of a double Fourier series, as is done for similar problems in Ref. 4. The contribution to this series from the terms with sufficiently large values of the axial and circumferential Fourier indices (m, n) can be ex-

Received July 14, 1967.

* Research Scientist, Aerospace Sciences Laboratory.

pressed as

$$\frac{4(1+\nu)\lambda}{\pi^2 Eh}\sum_{m,n\ \mathrm{large}}^{\infty}\left\{\frac{1}{m^2+n^2\lambda^2}\right\}\times\left\{1-\frac{(1+\nu)m^2}{2(m^2+\lambda^2n^2)}\right\}\cos^2(m\pi\bar{x}/L)$$

where $\lambda = L/\pi a$, which is greater than the corresponding sum

$$\frac{4(1+\nu)\lambda^3}{\pi^2 Eh}\sum_{m,n\ \mathrm{large}}^{\infty}\left\{\frac{n^2}{(m^2+n^2\lambda^2)^2}\right\}\cos^2(m\pi\bar{x}/L)$$

An application of the integral test to this latter sum reveals that it diverges. Thus the double Fourier series representation for $\hat{u}$ also diverges. One might have expected this since, for the problem of an in-plane force at a point of an infinite plate, as considered in Ref. 5, the displacement in the direction of the force at the point of application of the force is found to be infinite. Since $\hat{u}$ is infinite, the modal representation for $\hat{u}$ will thus diverge. For fixed ω_c, each one of the infinite number of terms, for which $\omega_j^2 \gg \omega_c^2$, in the modal representation for $\tilde{u}$ assumes the same form as the corresponding term in the divergent modal representation for $\hat{u}$. This then implies the divergence of the modal representation for $\tilde{u}$ and hence of

$$[u]_{\substack{\theta=0\\x=\bar{x}}}$$

These difficulties may be removed by distributing the attached mass over a small area of the shell.

References

[1] Ojalvo, I. U. and Newman, M., "Natural Vibrations of a Stiffened Pressurized Cylinder with an Attached Mass," *AIAA Journal*, Vol. 5, No. 6, June 1967, pp. 1139–1146.

[2] Bisplinghoff, R. L., Ashly, H., and Halfman, R. L., *Aeroelasticity*, Chap. X, Addison-Wesley, Reading, Mass., 1957, p. 635.

[3] Bushnell, D., "Dynamic Response of Two-Layered Cylindrical Shells to Time Dependent Loads," *AIAA Journal*, Vol. 3, No. 9, Sept. 1965, pp. 1698–1703.

[4] Bijlaard, P. P., "Stresses From Local Loadings in Cylindrical Pressure Vessels," *Transactions of the American Society of Mechanical Engineers*, Vol. 77, 1955, pp. 805–816.

[5] Timoshenko, S. and Goodier, J. N., *Theory of Elasticity*, 2nd ed., Chap. IV, McGraw-Hill, New York, 1951, p. 112.

Reply by Authors to R. D. Smith

I. U. Ojalvo* and M. Newman*
Harry Belock Associates Inc., Great Neck, N. Y.

THE preceding comments are unfortunate since they are either stated without proof (mathematically speaking), or loosely worded. In addition, the discusser has failed to appreciate a key point in our analysis, i.e., only the unweighted "breathing" modes are employed as trial functions for the weighted cylinder. His three main criticisms are summarized below and then negated individually: 1) The assumption of vanishing axial restraint employed in the analysis is not applicable to the ring-stiffened shells tested. 2) A basic discrepancy exists between the experimental results and the analytical predictions for cylinder 1. 3) The expression for axial displacement [Eq. (29) of the original paper[3]] diverges as the number of modal functions that employed ($\bar{n}$) becomes infinite.

Received September 22, 1967.
* Engineering Consultant.

Since the discusser has attempted a more quantitative argument to support item 3, we choose to deal with it first. He has contended that divergence of our for the

$$[u]_{\substack{\theta=0\\x=\bar{x}}}$$

weighted cylinder depends upon divergence of his next to last series. We agree that his series divergences, but actually it does not represent the series obtained when only the unweighted cylinder "breathing" modes are employed, as was stated in the paragraph following Eq. (22) of the subject paper.[3] Our approach is similar to that used successfully by Reissner[1] and later by Fung, Sechler, and Kaplan,[2] i.e., only the lowest root of the cubic frequency equation is considered for each nodal pattern. The other two roots are considerably higher than the smallest one and correspond to predominately tangential modes. It is demonstrated in Ref. 2 that for $k > 2$ and $a/h > 100$, the frequencies of the tangential modes are over 60 times larger than the corresponding radial ("breathing") frequencies.

Consistent with this simplification, the contribution to $\hat{u}$ for large wave numbers is actually (using discusser's notation)

$$\frac{(1-\nu^2)(1+\nu)^2\lambda^{11}}{2\pi^2Eh(\beta^2)(1+\beta)^2}\times\sum_{m,n\ \mathrm{large}}^{\infty}\left\{\frac{m^2n^4\cos^2(m\pi\bar{x}/L)}{[m^2+\lambda^2n^2/(1+\beta)](m^2+\lambda^2n^2)^6}\right\}$$

where $\beta = h^2/(12a^2)$.

This series is rapidly convergent and therefore obviates the need of 1) "distributing the attached mass over a small area of the shell," and 2) including an axial rigid body mode.

Turning to item 1, the discusser's statement that

$$|u|]_{\mathrm{adjacent\ bays}} \ll |u|]_{\mathrm{excited\ bay}}$$

which he has simply stated without proof, does not follow from the experimental observations that

$$|w|]_{\mathrm{adjacent\ bays}} \ll |w|]_{\mathrm{excited\ bay}}$$

Further, the experimental evidence for cylinders 1 through 4 inclusive (see Table 2 of our paper[3]) indicates that the axial forces were not sufficiently "appreciable" to affect the rather close agreement with our theoretical results. Thus, it appears that the method of test cylinder fabrication was effective in permitting axial warping across the intermediate rings and at the bulkhead joints.

With regard to item 2, where the discusser refers to ". . . forces exerted by the attached mass . . . act(ing) through displacements of zero amplitude, . . .," it should be evident that the addition of a concentrated mass at a nodal point does not produce any inertia force or "perturbation" of that uncoupled mode, since the mass is motionless. Therefore, a weighted system frequency can equal an unweighted frequency with the mass not driving the shell at resonance.

Finally, both the analytical and experimental evidence tend to contradict the notion that a mass addition merely causes a downward "perturbation" of the unweighted frequencies. Rather, it has been shown that surprisingly small weight additions can create a highly localized response, causing a drastic change in the fundamental mode-shape and a large reduction in fundamental frequency.

References

[1] Reissner, E., "On Transverse Vibration of Thin Shallow Elastic Shells," *Quarterly of Applied Mathematics*, Vol. 13, 1955, pp. 169–176.

[2] Fung, Y. C., Sechler, E. E., and Kaplan, A., "On the Vibration of Thin Cylindrical Shells under Internal Pressure," *Journal of Aeronautical Sciences*, Vol. 24, 1967, pp. 650–660.

[3] Ojalvo, I. U. and Newman, M., "Natural Vibrations of a Stiffened Pressurized Cylinder with an Attached Mass," *AIAA Journal*, Vol. 5, No. 6, June 1967, pp. 1139–1146.